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On the Saturation Problem for Strong Approximation by Double Fourier Series*

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1. INTRODUCTION

After Alexits [6], Leindler [6, 7], and Gogoladze [1] investigated estimates of strong approximation by Fourier series in 1965, G. Freud [8] raised the corresponding saturation problem in 1969. Study on this topic has since been carried on over a decade, but it seems that most of the results obtained are limited to the case of one dimension [9–11]. The saturation problem in two dimensions was considered in [2].

Let $C_{2\pi\times 2\pi}$ be the space of continuous functions with period 2π in each variable. For $f(x, y) \in C_{2\pi\times 2\pi}$, let $S_{mn}(f, x, y)$ denote the Fourier partial sums, and $E_{mn}(f) = \inf_{T_{mn}} \|f - T_{mn}\|_{C_{2\pi\times 2\pi}}$ the best uniform approximation of f by double trigonometric polynomials of order m, n. (We use $\|\cdot\|$ to denote $\|\cdot\|_{C_{2\pi\times 2\pi}}$ henceforth.)

We consider the rectangular strong approximation operator H^p_{MN} defined by

$$H^p_{MN}(f, x, y) = \left\{\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |S_{mn}(f, x, y) - f(x, y)|^p\right\}^{1/p},$$

and the Marcinkiewicz diagonal strong approximation operator h^p_{NN} defined by

$$h^p_{NN}(f, x, y) = \left\{\frac{1}{N+1} \sum_{n=0}^{N} |S_{nn}(f, x, y) - f(x, y)|^p\right\}^{1/p},$$

where $p > 0$.

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In [1, 4] the following estimates were obtained for these two operators.

$$\|H^p_{MN}(f)\| \leqslant C \left\{\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{h=0}^{N} E^p_{mn}(f)\right\}^{1/p},$$

$$\|h^p_{NN}(f)\| \leqslant C \left\{\frac{1}{N+1} \sum_{h=0}^{N} E^p_{nn}(f)\right\}^{1/p}.$$

(C is used to denote constants with different positive values at different places, independent of M, N, and f.) Of course, $\|H^p_{MN}\| \to 0$ $\|h^p_{NN}\| \to 0$ (as $M, N \to 0$), and the speed of convergence to zero depends on the smoothness of f. These two operators are saturable with saturation degree $(1/M + 1/N)^{1/p}$ and $(1/N)^{1/p}$, respectively [2].

If we call the collection of functions $\{f: \|H^p_{MN}(f)\| = \mathcal{O}[(1/M + 1/N)^{1/p}]\}$ the saturation class of H^p_{MN}, and $\{f: \|h^p_{NN}(f)\| = \mathcal{O}[(1/N)^{1/p}]\}$ the saturation class of h^p_{NN}, what can we say about functions in these saturation classes? We obtained the following results [2]:

If f is a saturation function, then the modulus of continuity of f has the order

$$\begin{aligned} \omega(f, t_1, t_2) &= O(t_1^{1/p} + t_2^{1/p}), && p > 1, \\ &= O\left(t_1 \log \frac{1}{t_1} + t_2 \log \frac{1}{t_2}\right), && p = 1, \\ &= O(t_1 + t_2), && 0 < p < 1, \end{aligned}$$

as t_1 and t_2 approaches zero. Here

$$\omega(f, t_1, t_2) = \sup_{\substack{|x_1 - x_2| \leqslant t_1, |y_1 - y_2| \leqslant t_2 \\ x_1, x_2 \in [0, 2\pi],\, y_1, y_2 \in [0, 2\pi]}} |f(x_1, y_1) - f(x_2, y_2)|$$

is the modulus of continuity of f.

The purpose of this paper is to analyze whether these results can be improved. In Theorem 1, we show that the above results cannot be improved in the sense of modulus of continuity. In Theorem 2 we discuss differentiability properties of functions in saturation classes for $0 < p < 1$, and finally in Theorem 3 we prove that these results also cannot be improved.

2. Main Results

THEOREM 1. *For each $p > 0$, there exists a function $f_p(x, y)$ with the properties*

$$\left.\begin{aligned} \|H^p_{MN}(f_p)\| &= O\left[\left(\frac{1}{M}+\frac{1}{N}\right)^{1/p}\right] \\ \|h^p_{NN}(f_p)\| &= O\left[\left(\frac{1}{N}\right)^{1/p}\right] \end{aligned}\right\} \quad (\text{as } M, N \to \infty),$$

and

$$\begin{aligned} \omega\left(f_p, \frac{1}{2^m}, \frac{1}{2^n}\right) &\geqslant C\left[\left(\frac{1}{2^m}\right)^{1/p} + \left(\frac{1}{2^n}\right)^{1/p}\right], && \text{if } \; p > 1, \\ &\geqslant C\left[\frac{m}{2^m} + \frac{n}{2^n}\right], && \text{if } \; p = 1, \\ &\geqslant C\left[\frac{1}{2^m} + \frac{1}{2^n}\right], && \text{if } \; 0 < p < 1, \end{aligned}$$

provided $m, n \geqslant 4$.

THEOREM 2. *Let $0 < p < 1$, and $1/p = r + \alpha$, where $r > 0$ is an integer and $0 < \alpha \leqslant 1$. If $f(x, y)$ is a saturation function of H^p_{NN} (or h^p_{NN}), that is, if*

$$\|H^p_{MN}(f)\| = O\left[\left(\frac{1}{M}+\frac{1}{N}\right)^{1/p}\right] \quad \text{as} \quad M, N \to \infty,$$

$$\left(\text{or} \quad \|h^p_{NN}(f)\| = O\left[\left(\frac{1}{N}\right)^{1/p}\right]\right).$$

Then the partial derivatives $\partial^r f/\partial x^{r_1}\, \partial y^{r_2}$ (r_1, $r_2 \geqslant 0$, $r_1 + r_2 = r$) exist and satisfy for $t_1, t_2 \to 0$

$$\begin{aligned} \omega\left(\frac{\partial^r f}{\partial x^{r_1}\, \partial y^{r_2}}, t_1, t_2\right) &= O(t_1^\alpha + t_2^\alpha), && \alpha \neq 1, \\ &= O\left(t_1 \log\frac{1}{t_1} + t_2 \log\frac{1}{t_2}\right), && \alpha = 1. \end{aligned}$$

THEOREM 3. *Let $0 < p < 1$, and $1/p = r + \alpha$, $0 < \alpha \leqslant 1$. If r is even, then there is a function $f_p(x, y)$, such that*

$$\|H^p_{MN}(f_p)\| = O\left[\left(\frac{1}{M}+\frac{1}{N}\right)^{1/p}\right],$$

and

$$\|h_{NN}^{p}(f_p)\| = O\left[\left(\frac{1}{N}\right)^{1/p}\right],$$

but for $m \geqslant 4$,

$$\omega\left(\frac{\partial^r f}{\partial x}, \frac{\pi}{2^m}, \frac{\pi}{2^m}\right) \geqslant C\left(\frac{1}{2^m}\right)^{\alpha}, \qquad \alpha \neq 1,$$

$$\geqslant C\left(\frac{m}{2^m}\right), \qquad \alpha = 1,$$

$$\omega\left(\frac{\partial^r f}{\partial y}, \frac{\pi}{2^m}, \frac{\pi}{2^m}\right) \geqslant C\left(\frac{1}{2^m}\right)^{\alpha}, \qquad \alpha \neq 1,$$

$$\geqslant C\left(\frac{m}{2^m}\right), \qquad \alpha = 1.$$

3. Auxiliary Lemmas

Lemma 1 is for the proof of Lemma 2. Lemmas 2 and 3 are preparations to prove Theorem 2.

Lemma 1. *For any* $0 < p < 1$ *we have*

$$E_{mm}(f)\left[\frac{E_{2m2m}(f)}{E_{mm}(f)}\right]^{1/p^4} \leqslant C\left\|\frac{1}{(m+1)^2}\sum_{\nu=m}^{2m}\sum_{\mu=m}^{2m}|S_{\nu\mu}-f|^p\right\|^{1/p}, \quad (3.1)$$

and hence, if f *is a saturation function of* H_{MN}^p,

$$E_{mm}(f)\left[\frac{E_{2m2m}(f)}{E_{mm}(f)}\right]^{1/p^4} \leqslant C\left[\left(\frac{1}{m}\right)^{1/p}\right]. \quad (3.2)$$

Proof. Employing the fact $E_{2m2m}(f) \leqslant \|(1/(m+1)^2)\sum_{\nu=m}^{2m}\sum_{\mu=m}^{2m} [S_{\nu\mu}-f]^{p^4+1-p^4}\|$ using Hölder's inequality twice for conjugate numbers $(1/p^2, 1/(1-p^2))$ and $(1/p, 1/(1-p))$, and noticing $\|(1/(m+1)(n+1)) \sum_{\nu=m}^{2m}\sum_{\mu=m}^{2m}|S_{\nu\mu}-f|^q\|^{1/q} \leqslant \Rightarrow CE_{mn}$, we obtain

$$E_{2m2m}(f)$$

$$\leqslant \frac{C}{(m+1)^2}\left\|\left[\sum_{\nu=m}^{2m}\sum_{\mu=m}^{2m}|S_{\nu\mu}-f|^{p^2}\right]^{p^2}\left[\sum_{\nu=m}^{2m}\sum_{\mu=m}^{2m}|S_{\nu\mu}-f|^{1+p^2}\right]^{1-p^2}\right\|$$

$$\leqslant \frac{C}{(m+1)^2} \left\| \left[\sum_{\nu=m}^{2m} \sum_{\mu=m}^{2m} |S_{\nu\mu} - f|^p \right]^{p^3} (m^2)^{(1-p)p^2} [(m^2)^{1-p^2} (E_{mm}^{1+p^2}(f))^{1-p^2}] \right.$$

$$\leqslant C E_{mm}^{1-p^4}(f) \left\| \frac{1}{(m+1)^2} \sum_{\nu=m}^{2m} \sum_{\mu=m}^{2m} |S_{\nu\mu} - f|^p \right\|^{p^3}.$$

This is equivalent to the statement of the lemma.

LEMMA 1′. *For* $0 < p < 1$, *we have*

$$E_m(f) \left[\frac{E_{2m2m}(f)}{E_{mm}(f)} \right]^{1/p^4} \leqslant C \left\| \frac{1}{m+1} \sum_{\nu=m}^{2m} |S_{\nu\nu} - f|^p \right\|^{1/p}$$

and hence, if f *is a saturation function of* h_{NN}^p, *we have the inequality*

$$E_{mm}(f) \left[\frac{E_{2m2m}(f)}{E_{mm}(f)} \right]^{1/p^4} \leqslant C \left(\frac{1}{m} \right)^{1/p}.$$

This result follows if one uses the estimate $E_{2m2m}(f) \leqslant \| 1/(m+1) \sum_{\nu=m}^{2m} [S_{\nu\nu} - f] \|$, and employs the method of the proof of Lemma 1.

LEMMA 2. *If* f *belongs to the saturation class of* H_{MN}^p (*or of* h_{NN}^p), *then*

$$E_{nn}(f) \leqslant C \left(\frac{1}{n} \right)^{1/p}. \tag{3.3}$$

Proof. For any m, from (3.2)

$$E_{2^m 2^m}(f) \left\{ \frac{E_{2^{m+1} 2^{m+1}}(f)}{E_{2^m 2^m}(f)} \right\}^{1/p^4} \leqslant C \left(\frac{1}{2^m} \right)^{1/p}.$$

We divide integers m into two classes. The number m belongs to the first class, N_1, if

$$\frac{E_{2^{m+1} 2^{m+1}}(f)}{E_{2^m 2^m}(f)} \leqslant \left(\frac{1}{2} \right)^{1/p}, \tag{3.4}$$

and otherwise belongs to the second class, N_2. The integers m of N_1 form some disjoint (perhaps infinite) intervals (m_1, m_1^*), (m_2, m_2^*),....

For every m in N_2, $\{E_{2^m 2^m}(f)/E_{2^{m+1} 2^{m+1}}(f)\}^{1/p^4} < q^{1/p^5}$, hence from (3.2)

$$E_{2^m 2^m}(f) \leqslant C 2^{1/p^5} \left(\frac{1}{2^m} \right)^{1/p}. \tag{3.5}$$

Next, let m_i be the first integer of one of the intervals of N_1. Then $m_i = 1$ is in N_2(if $m_i = 1$, then $E_{22} \leqslant E_{11} \leqslant (2^{1/p}E_{11})(1/2)^{1/p} = C(1/2)^{1/p}$). For every m in $[m_i, m_i^*)$, by (3.4) and (3.5),

$$E_{2^m 2^m}(f) \leqslant \left(\frac{1}{2^{m-m_i}}\right)^{1/p} E_{2^{m_i}2^{m_i}}(f) \leqslant \left(\frac{1}{2^{m-m_i}}\right)^{1/p} E_{2^{m_i-1}2^{m_i-1}}$$

$$\leqslant C\left(\frac{1}{2^{m-m_i}}\right)^{1/p} q^{1/p^5}\left(\frac{1}{2^{m_i-1}}\right)^{1/p} = C2^{1/p+1/p^5}\left(\frac{1}{2^m}\right)^{1/p}.$$

Summarizing, $E_{2^m 2^m}(f) \leqslant C(1/2^m)^{1/p}$. We have proved (3.3) if $n = 2^m$. For an arbitrary n, let $2^m \leqslant n < 2^{m+1}$, then

$$E_{nn}(f) \leqslant E_{2^m 2^m}(f) \leqslant C\left(\frac{1}{2^m}\right)^{1/p} \leqslant C\left(\frac{2}{n}\right)^{1/p} = C\left(\frac{1}{n}\right)^{1/p}.$$

This proves Lemma 2.

LEMMA 3. *Let*

$$V_{mm}(f, x, y) = \frac{1}{m+1}\sum_{v=m}^{2m} S_{vv}(f, x, y), \tag{3.6}$$

and

$$U_{mm}(f, x, y) = V_{2^m 2^m}(x, y) - V_{2^{m-1}2^{m-1}}(f, x, y).$$

Then we have

$$\|U_{mm}(f, x, y)\| = O(E_{2^m 2^m}). \tag{3.7}$$

Proof.

$$\begin{aligned} U_{mm}(f, x, y)\| &\leqslant \|V_{2^m 2^m}(f, x, y) - f(x, y)\| \\ &\quad + \|V_{2^{m-1}2^{m-1}}(f, x, y) - f(x, y)\|, \\ &\leqslant \left\|\frac{1}{2^m+1}\sum_{v=2^m}^{2^{m+1}} |S_{vv} - f|\right\| + \left\|\frac{1}{2^{m-1}+1}\sum_{v=2^{m-1}}^{2^m} |S_{vv} - f|\right\| \\ &\leqslant \left\|\frac{1}{2^m+1}\sum_{v=2^m}^{2^{m+1}} [|f - T^*_{2^{m+1}2^{m+1}}| + |S_{vv}(f - T^*_{2^{m+1}2^{m+1}})|]\right. \\ &\quad \left. + \frac{1}{2^{m-1}+1}\sum_{v=2^{m-1}}^{2^m} [|f - T^*_{2^m 2^m}| + |S_{vv}(f - T^*_{2^m 2^m})|]\right\|, \end{aligned}$$

where the T^*_{ij}'s are the trigonometric polynomials of best approximation of orders i, j. In [3] I have proved the inequality $\| \sum_{v=k}^{2k} |S_{vv}(g)| \| \leqslant (k+1) \| g \|$. This implies

$$\| U_{mm}(f, x, y) \| \leqslant C E_{2^m 2^m}.$$

4. Proofs of the Theorems

1. *Proof of Theorem* 1. Let

$$f_p(x, y) = g_p(x) + g_p(y), \; g_p(x) = \sum_{v=1}^{\infty} \frac{\sin vx}{v^{1+(1/p)}}.$$

It is known [9] that the function $g_p(x)$ has the properties

$$F(g_p, x) := \sum_{m=1}^{\infty} |S_m(g_p, x) - g_p(x)|^p \leqslant \text{const. and, for } 0 < t < 1/2^4$$

$$\begin{aligned} \omega(g_p, t) &\geqslant c t^{1/p}, && p > 1, \\ &\geqslant ct \log \frac{1}{t}, && p = 1, \\ &\geqslant ct, && 0 < p < 1. \end{aligned}$$

It follows from the above,

$$\begin{aligned} H_{MN}^p(f_p, x, y) &\leqslant H_{MN}^p(g_p, x) + H_{MN}^p(g_p, y) \\ &\leqslant 2 \left[\frac{1}{M} \cdot F(g_p, x) + \frac{1}{N} F(g_p, y) \right]^{1/p} \\ &\leqslant C \left(\frac{1}{M} + \frac{1}{N} \right)^{1/p}. \end{aligned}$$

For the same reason

$$h_{NN}^p(f_p, x, y) \leqslant C \left(\frac{1}{N} \right)^{1/p}.$$

But since

$$\begin{aligned} \omega(f_p, t_1, t_2) &\geqslant \max\{\omega(f_p, t_1, 0), \omega(f_p, 0, t_2)\} \\ &\geqslant \tfrac{1}{2}\{\omega(f_p, t_1, 0) + \omega(f_p, 0, t_2)\} \\ &\geqslant \tfrac{1}{2}\{\omega(g_p, t_1) + \omega(g_p, t_2)\}, \end{aligned}$$

we have

$$\omega\left(f_p,\frac{1}{2^m},\frac{1}{2^n}\right)\geqslant C\left\{\left(\frac{1}{2^m}\right)^{1/p}+\left(\frac{1}{2^n}\right)^{1/p}\right\},\qquad p>1,$$
$$\geqslant C\left\{\frac{m}{2^m}+\frac{n}{2^n}\right\},\qquad p=1,$$
$$\geqslant C\left\{\frac{1}{2^m}+\frac{1}{2^n}\right\},\qquad 0<p<1,$$

when $m, n \geqslant 4$. Theorem 1 is established.

2. *Proof of Theorem* 2. With V_{mm} defined by (3.6) $U_{mm}(f, x, y) = V_{2^m 2^m}(f, x, y) - V_{2^{n-1}2^{m-1}}(f, x, y)$. From $\| V_{2^m 2^m}(f, x, y) - f(x, y)\| = \| 1/(2^m + 1) \sum_{\nu=2^m}^{2^{m+1}} | S_{\nu\nu}(f, x, y) - f(x, y)| \| \leqslant 2/(2^{m+1} + 1) \sum_{\nu=0}^{2^{m+1}} E_{\nu\nu}(f) \to 0$ (for $m \to \infty$), it follows that $V_{2^m 2^m}(f, x, y) \to f(x, y)$ uniformly. Hence, $f(x, y) = \sum_{m=0}^{\infty} U_{mm}(f, x, y)$, where the series converges uniformly. Because of Bernstein's inequality and Lemma 3, we have

$$\left|\frac{\partial^r U_{mm}}{\partial x^{r_1}\,\partial y^{r_2}}(x, y)\right| \leqslant 2^{mr}\,\| U_{mm}(x, y)\| \leqslant c2^{mr}E_{2^m 2^m}.$$

By Lemma 2, the numerical series $\sum_{m=0}^{\infty} 2^{mr}E_{2^m 2^m} \leqslant C \sum_{m=0}^{\infty} 2^{mr}(1/2^m)^{1/p}$ converges. Therefore $(\partial^r f)/(\partial x^{r_1}\,\partial y^{r_2})(x, y)$ exists, and $\sum_{m=0}^{\infty}(\partial^r U_m)/(\partial x^{r_1}\,\partial y^{r_2})(x, y)$ converges uniformly to $(\partial^r f)/(\partial x^{r_1}\,\partial y^{r_2})(x, y)$.

Now we try to estimate the modulus of continuity of $(\partial^r f)/(\partial x^{r_1}\,\partial y^{r_2})$. For any $0 < t < \pi/2^n$, from Bernstein's inequality, Lemmas 3 and 2, we have

$$\left|\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}(x+t, y) - \frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}(x, y)\right|$$
$$\leqslant \sum_{m=0}^{n} t\left\|\frac{\partial^{r+1} U_{mm}}{\partial x^{r_1+1}\,\partial y^{r_2}}\right\| + 2\sum_{m=n+1}^{\infty}\left\|\frac{\partial^r U_{mm}}{\partial x^{r_1}\,\partial y^{r_2}}\right\|$$
$$\leqslant C\left\{\frac{1}{2^n}\sum_{m=0}^{n} 2^{m(r+1)}E_{2^m 2^m} + 2\sum_{m=n+1}^{\infty} 2^{mr}E_{2^m 2^m}\right\}$$
$$\leqslant C\left(\frac{1}{2^n}\right)^{\alpha},\qquad \alpha \neq 1,$$
$$\leqslant C\,\frac{n}{2^n},\qquad \alpha = 1.$$

Therefore

$$\omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, \frac{\pi}{2^n}, 0\right) = O\left[\left(\frac{1}{2^n}\right)^{\alpha}\right], \qquad \alpha \neq 1,$$

$$= O\left[\frac{n}{2^n}\right], \qquad \alpha = 1.$$

From the property of modulus of continuity, we have

$$\omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, t_1, 0\right) = O(t_1^{\alpha}), \qquad \alpha \neq 1,$$

$$= O\left(t_1 \log \frac{1}{t_1}\right), \qquad \alpha = 1.$$

Similarly,

$$\omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, 0, t_2\right) = O(t_2^{\alpha}), \qquad \alpha \neq 1,$$

$$= O\left(t \log \frac{1}{t_2}\right), \qquad \alpha = 1,$$

and then,

$$\omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, t_1, t_2\right) \leqslant \omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, t_1, 0\right) + \omega\left(\frac{\partial^r f}{\partial x^{r_1}\,\partial y^{r_2}}, 0, t_2\right)$$

$$= O(t_1^{\alpha} + t_2^{\alpha}), \qquad \alpha \neq 1,$$

$$= O\left(t_1 \log \frac{1}{t_1} + t_2 \log \frac{1}{t_2}\right), \qquad \alpha = 1.$$

This is the result we required in Theorem 2.

3. *Proof of Theorem* 3. Assume

$$f_p(x, y) = \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu^{1+1/p}} + \sum_{\mu=1}^{\infty} \frac{\sin \mu y}{\mu^{1+1/p}},$$

since

$$\frac{\partial^r f_p}{\partial x^r}(x, y) = \pm \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu^{\alpha+1}}, \frac{\partial^r f_p}{\partial y^r}(x, y) = \pm \sum_{\mu=1}^{\infty} \frac{\sin \mu y}{\mu^{\alpha+1}},$$

using the proof of Theorem 1, we see

$$\omega\left(\frac{\partial^r f_p}{\partial x^r}, \frac{1}{2^m}, \frac{1}{2^m}\right) \geqslant \omega\left(\frac{\partial^r f_p}{\partial x^r}, \frac{1}{2^m}, 0\right)$$

$$\geqslant C\left(\frac{1}{2^m}\right)^{\alpha}, \qquad \alpha \neq 1,$$

$$\geqslant C\frac{m}{2^m}, \qquad \alpha = 1,$$

$$\omega\left(\frac{\partial^r f_p}{\partial y^p}, \frac{1}{2^m}, \frac{1}{2^m}\right) \geqslant C\left(\frac{1}{2^m}\right)^{\alpha}, \qquad \alpha \neq 1,$$

$$\geqslant C\frac{m}{2^m}, \qquad \alpha = 1,$$

when $m \geqslant 4$, and

$$\|H^p_{MN}(f_p)\| = O\left[\left(\frac{1}{M}+\frac{1}{N}\right)^{1/p}\right],$$

$$\|h^p_{NN}(f_p)\| = O\left[\left(\frac{1}{N}\right)^{1/p}\right].$$

Theorem 3 is established.

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